SAPIENZA, UNIVERSITÁ DI ROMA
DIP. DI FISICA

Written Exam #2 2020/2021

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *16 July 2021*

Exercise 1

A hypothetical spherical galaxy has an internal bolometric luminosity volume density distribution expressed by the function

$$I(r) = \frac{I_0}{1 + \left(\frac{r}{r_c}\right)^2}, \tag{1}$$

where $r \geq 0$, and I_0 and r_c are positive constants (physical dimensions of I_0 are $\mathrm{ML^{-1}T^{-3}}$).

It is asked:

Questions

1. to give the explicit expression for the total luminosity of the galaxy, assuming a galaxy cutoff radius R;
2. to give the explicit expression for the projected surface luminosity distribution $\sigma(s)$ where s is the distance to the galactic center on the plane of the projection;
3. assuming a constant mass-to-light ratio, $M/L = 10(M/L)_\odot$, to give the explicit expression of the circular velocity $v_c(r)$ at $r = r_c$.

Provide also numerical values for the answers to questions 1 (in solar luminosities, $\mathrm{L}_\odot$) and 3 (in km s^{-1}) assuming $r_c = 1$ kpc, $R/r_c = 50$, $I_0 = 1\ \mathrm{L}_\odot\ \mathrm{pc}^{-3}$.

Note: The symbol $\odot$ refers to the Sun. $\mathrm{L}_\odot \simeq 3.85 \times 10^{33}$ erg s^{-1}. Newton's gravitational constant $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}; 1 pc $\simeq 3.09 \times 10^{16}$ m.

Exercise 2

A spherical galaxy, whose age is $\tau \simeq 13$ Gyr, radius R, and total mass M, is deprived of dark matter and composed only by $N_* > 10^{11}$ stars uniformly distributed. When, for computational necessity, the galaxy is sampled with a number $N < N_*$ of objects, it is asked:

Question

to determine the condition which gives the threshold in N below which the sampling is surely too poor to respect the non-collisionality of the system.

Answers

Exercise 1

1. A: The total luminosity is given by the integral

$$L_{tot} = 4\pi I_0 \int_0^R \frac{r^2}{1+\left(\frac{r}{r_c}\right)^2}\, dr = 4\pi L_0 r_c^3 \int_0^{R/r_c} \frac{x^2}{1+x^2}\, dx = \tag{2}$$

$$= 4\pi I_0 r_c^3 \left(\int_0^{R/r_c} dx - \int_0^{R/r_c} \frac{1}{1+x^2}\, dx \right) = 4\pi I_0 r_c^3 \left(\frac{R}{r_c} - \arctan \frac{R}{r_c} \right). \tag{3}$$

Numerically, with the given values for I_0, r_c, R/r_c it results $L_{tot} \simeq 6.09 \times 10^{11} \mathrm{L}_\odot$.

2. A: Given the generic point (x, y) on the projected (orthogonal to the line of sight) circular disc of the galaxy whose distance to the center is $s = \sqrt{x^2+y^2}$ and considering an axis z orthogonal to the disc, passing by (x, y) and parallel to the line of sight the expression of the projected (surface) absolute luminosity is

$$\sigma(s) = \int_{-z_M}^{z_M} I(z)\, dz = 2\int_0^{z_M} I(z)\, dz, \tag{4}$$

where $-z_M$ and z_M are the minimum and maximum possible values of z, i.e. the negative and positive intersection of the z axis with the galactic surface ($r = R$). Due to that $z^2 + s^2 = r^2$, the given expression of $L(r)$ leads to that the integral above rewrites as

$$\sigma(s) = 2I_0 \int_0^{z_M} \frac{dz}{1+\left(\frac{\sqrt{z^2+s^2}}{r_c}\right)^2}. \tag{5}$$

Letting $a^2 = 1 + (s/r_c)^2$ and by the substitution $x = z/r_c$, the integral above is

$$\sigma(s) = 2I_0 r_c \int_0^{z_M/r_c} \frac{dx}{a^2+\left(\frac{z}{r_c}\right)^2} = 2\frac{I_0 r_c}{a} \arctan \frac{z_M}{r_c a} = 2\frac{I_0 r_c}{\sqrt{1+(s/r_c)^2}} \arctan \frac{z_M/r_c}{\sqrt{1+(s/r_c)^2}}. \tag{6}$$

3. A: The circular velocity is

$$v_c(r) = \sqrt{G\frac{M(r)}{r}}. \tag{7}$$

In our case $M(r) = (M/L)4\pi I_0\, r_c^3\,(r/r_c - \arctan r/r_c)$, where M/L is the mass-to-light ratio (assumed =10). Consequently

$$v_c(r) = \sqrt{G4\pi(M/L)I_0\, r_c^2}\,\sqrt{\frac{r/r_c - \arctan r/r_c}{r/r_c}}, \tag{8}$$

to give $v_c(r_c) = \sqrt{G4\pi(M/L)I_0\, r_c^2}\,\sqrt{1-\pi/4}$.

Numerically, with the given values of G, (M/L), I_0 and r_c it is $v_c(r_c) \simeq 346$ km s^1.

Exercise 2

A: The formula for the 2-body relaxation time gives $t_{rel} = N/(6\ln N)t_{cr}$, where t_{cr} is the system crossing time which for a uniform sphere is $t_{cr} = R^{3/2}/\sqrt{(3/5)GM/R}$. So, the condition to be fulfilled in order the sampling to be surely insufficient is

$$t_{rel} = \frac{1}{6}\frac{N}{\ln N}\frac{R^{3/2}}{\sqrt{\frac{3}{5}\frac{GM}{R}}} \le \tau, \tag{9}$$

that reflects into this condition for N

$$\frac{N}{\ln N} \le \frac{6\sqrt{\frac{3}{5}\frac{GM}{R}}}{R^{3/2}}\tau, \tag{10}$$

which is surely satisfied for $N \le \overline{N}$ where $\overline{N}$ is the integer part of the solution of the non-linear equation

$$f(x) = \frac{x}{\ln x} - \frac{6\sqrt{\frac{3}{5}\frac{GM}{R}}}{R^{3/2}}\tau = 0. \tag{11}$$

Such solution exists unique because $x \ge 1$, $\lim_{x\to\infty} = +\infty$ and the derivative

$$f'(x) = \frac{\ln x - 1}{\ln^2 x} \tag{12}$$

is > 0 for $x > e$ and so for every $N > 3$ the function is monotonically increasing.

Sapienza, Universitá di Roma
Dip. di Fisica

Written Exam #3 2020/2021

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *13 September 2021*

Exercise 1

A particle moves in the infinitely extended mass density radial distribution

$$\rho(r) = \rho_0 \left(\frac{r}{R}\right)^{-1}, \tag{1}$$

where ρ_0 and R are two positive constants. Assuming $r(0) = R$, $\dot{r}(0) = 0$ and $\dot{\theta}(0) = 0$ as initial conditions, it is asked

Questions

(i) to determine the time needed to reach the origin,

and

(ii) to determine the density, ρ_1, of a homogeneous sphere in which the particle of same initial conditions reaches the center in the same time of the previous case.

Exercise 2

A particle of unitary mass moves radially in a central, attractive, force field deriving by a potential $U(r)$. The potential $U(r)$ is continuous for every $r \geq 0$ and also characterized by: $U(r) > 0$, $\lim_{r \to +\infty} U(r) = 0$.

It is asked:

Question

to give a lower boundary for the period of motion of particles of energy E and apocenter distance r_+.

Answers

Exercise 1

Answer to question (i)

The mass within generic radius r is

$$M(r) = 4\pi\rho_0 R^3 \int_0^{r/R} x\,dx = 2\pi\rho_0 R r^2, \tag{2}$$

which implies an attractive, constant gravitational force field

$$\mathbf{F}(r) = -G\frac{M(r)}{r^2}\mathbf{e}_r = -G\frac{2\pi\rho_0 R r^2}{r^2}\mathbf{e}_r = -G2\pi\rho_0 R\mathbf{e}_r \tag{3}$$

If $\dot{\theta}(0) = 0$ the motion is purely radial so that the equation of motion in such a force field reduces to

$$\ddot{r} = -G2\pi\rho_0 R \tag{4}$$

easily solved in

$$r(t) = r(0) + \dot{r}(0)t - G\pi\rho_0 R t^2, \tag{5}$$

that gives, for the assumed c.i., $r(t) = R - G\pi\rho_0 R t^2$. The time T needed to reach the center is thus obtained by letting $r(T) = 0$ in this relation and solving for $t = T$

$$T = \frac{1}{\sqrt{G\pi\rho_0}} = \frac{\sqrt{G\pi\rho_0}}{G\pi\rho_0}, \tag{6}$$

independent of R.
Another way to get same result is by computing the integral

$$T = \int_R^0 \frac{dr}{\dot{r}} = \frac{1}{\sqrt{2}}\int_0^R \frac{dr}{\sqrt{E+U}}, \tag{7}$$

where the potential is $U(r) = -G2\pi\rho_0 R r + c$ with c constant of integration. Given that $E = E_0 = \frac{1}{2}\dot{r}_0^2 - U(r_0) = G2\pi\rho_0 R^2 - c$, the above integral writes as

$$T = \frac{1}{\sqrt{G4\pi\rho_0 R}} \int_0^R \frac{dr}{\sqrt{R-r}} = \frac{1}{\sqrt{G\pi\rho_0}}. \tag{8}$$

Answer to question (ii)

In a homogeneous sphere of constant mass density ρ_1 the time to reach the center is independent of the initial position $r(0)$ when $\dot{r}(0) = 0$ and it is $T_1 = P/4$ where p is the period $P = 2\pi/\omega$ where $\omega^2 = 2\pi G\rho_1$. So the required condition $T_1 = T$ gives

$$\frac{1}{2}\frac{\pi}{\sqrt{2\pi G\rho_1}} = \frac{1}{\sqrt{G\pi\rho_0}}, \tag{9}$$

which solved for ρ_1 gives $\rho_1 = \frac{\pi^2}{8}\rho_0$.

Exercise 2

A: The period of motion is, in the hypothesis of purely radial motion

$$T_r = 2\int_0^{r_+} \frac{dr}{\sqrt{2\left(E - V_eff\right)}}, \tag{10}$$

where r_+ is the apocenter distance (root of the equation $E - V_{eff} = 0$. Given that $U(r)$ is attractive and continuous ($U'(r) < 0 \; \forall r > 0$) means that its positive central value $U(0)$ is its maximum. Consequently

$$T_r = 2\int_0^{r_+} \frac{dr}{\sqrt{2\left(E - V_eff\right)}} = 2\int_0^{r_+} \frac{dr}{\sqrt{2\left(E + U(r)\right)}} \geq \sqrt{2}\int_0^{r_+} \frac{dr}{\sqrt{\left(E + U(0)\right)}} = \tag{11}$$

$$= \sqrt{2}\frac{r_+}{\sqrt{\left(E + U(0)\right)}}. \tag{12}$$

SAPIENZA, UNIVERSITÁ DI ROMA
DIP. DI FISICA

Written Exam #4 2020/2021

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *9 November 2021*

Exercise 1

A particle of mass m is subjected to a force field

$$\mathbf{F} = -\frac{k}{r^{3+\alpha}}\,\mathbf{e}_r, \tag{1}$$

where k is a positive constant, α a parameter and $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the radial direction.
Given as initial position $\mathbf{r}_0 = x_0\,\mathbf{i}$ (with $x_0 > 0$) and as initial velocity $\mathbf{v}_0$ at angle β with the positive x axis, it is asked

Questions

(i) to show that the equation of motion for the radial coordinate can be written as

$$\ddot{r} = -\frac{1}{mr^3}\left(\frac{k}{r^\alpha} - m\,x_0^2\,v_0^2\sin^2\beta\right), \tag{2}$$

and

(ii) to show upon what conditions on α the force field in eq. (1) is compatible with a gravitational origin.

Exercise 2

Let us consider the spherical matter density law

$$\rho(r,t) = \begin{cases} \rho_0\left(\dfrac{r}{R(t)}\right)^2, & \text{for } r \le R(t), \\ 0, & \text{for } r > R(t), \end{cases} \tag{3}$$

where $\rho_0 > 0$ and $R(t)$ is a positive, regular, function of time. Assuming for the above density distribution the generalization in integral form of the moment of inertia, I, of N particles respect to the origin, it is asked

Question

to write the differential equation to be satisfied by $R(t)$ in order to have a virialized matter distribution.

Answers

Exercise 1

Answer to question (i)

It is sufficient to add to F/m the contribution from the centrifugal by writing L_z^2 as function of x_0, v_0, β.

Answer to question (ii)

In order to have $\rho > 0$ (by mean of the Poisson's equation it is needed that

$$\rho(r) = -\frac{1}{4\pi G}\frac{d}{dr}\left[r^2\left(-\frac{k}{m}\frac{1}{r^{3+\alpha}}\right)\right] = -(1+\alpha)\frac{k}{4\pi G m}\, r^{-(2+\alpha)} > 0, \tag{4}$$

that requires $\alpha < -1$.

Exercise 2

The moment of inertia writes as

$$I(t) = \int_0^R r^2\, dm = 4\pi\, R^5 \rho_0 \int_0^1 x^6\, dx = \frac{4\pi\rho_0 R^5}{7}, \tag{5}$$

so that

$$\ddot{I} = \frac{20\pi\rho_0}{7}\left[4R^3\dot{R}^2 + R^4\ddot{R}\right], \tag{6}$$

so the virial conditions $\ddot{I} = 0$ writes as

$$4\dot{R}^2 + R\ddot{R} = 0. \tag{7}$$

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Written Exam #5 2020/2021

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *22 February 2022*

Exercise 1

Given the one-parameter family of positive spherically symmetric potentials $U(r;c)$ such to satisfy the differential equation

$$U' + \left(\frac{2}{r} - 1\right) U = 0, \tag{1}$$

where the apex stands for derivative respect to r, it is asked

Questions

(i) to determine the regions of attractivity and repulsivity of the potential;

and

(ii) to evaluate the orbital energy of the particle of unitary mass and unitary angular momentum starting from $r(0) = 1/2$ with zero radial velocity, knowing that the absolute value of the circular velocity of the unitary mass particle at distance $r = 1$ from the origin is $v_c(1) = \sqrt{2e}$ (e is the Euler's number).

Exercise 2

Two gaseous, singular, isothermal spheres in gravitational equilibrium of identical chemical composition are characterized by their (constant) temperatures T_1 and T_2.

Questions

(i) Find the ratio of the radii of the two spheres that contain the same arbitrarily given quantity of mass, M,

and

(ii) determine the ratio of the period of circular orbits of given radius R for the two cases.

Answers

Exercise 1

Answer to question (i)

The potential $U(r)$ is given by

$$\int \frac{dU}{U} = -\int \left(\frac{2}{r} - 1\right) dr, \tag{2}$$

that leads to

$$U(r) = e^c \frac{e^r}{r^2}, \tag{3}$$

where c is an integration constant. Consequently, the regions of attractivity and repulsivity are given by

$$U' = -\left(\frac{2}{r} - 1\right) e^c \frac{e^r}{r^2} \leq 0 \Leftrightarrow r \leq 2 \quad \text{(attractivity)}, \tag{4}$$

$$U' = -\left(\frac{2}{r} - 1\right) e^c \frac{e^r}{r^2} \geq 0 \Leftrightarrow r \geq 2 \quad \text{(repulsivity)}. \tag{5}$$

Note that the answer to (i) does not necessarily require the explicit knowledge of the potential because the simple knowledge $U > 0$ leads to the answer by using the relation in Eq. 1. But the knowledge of the expression of $U(r)$ is needed to. answer to question (ii).

Answer to question (ii)

Letting $r_0 = 1/2$, $\dot{r}_0 = 0$ and $L_0 = 1$ the searched energy is

$$E_0 = \frac{1}{2}\frac{L_0^2}{r_0^2} - U(r_0) = 2 - 4e^c\sqrt{e}. \tag{6}$$

The value of the constant c in the potential is determined by the condition on the circular velocity. The generic circular velocity is

$$v_c(r) = \sqrt{r|U'|}, \tag{7}$$

which computed for $r = 1$ gives $v_c(1) = \sqrt{e^c e}$. The condition $v_c(1) = 1$ leads to $c = -1$ so that $E_0 = 2 - 4/\sqrt{e} \simeq -0,426122639$.

Exercise 2

The singular isothermal sphere profiles are

$$\rho_i(r) = \frac{kT_i}{2\pi G\langle m\rangle} r^{-2}, \tag{8}$$

where k is the Boltzmann's constant, $\langle m\rangle$ is the average gas particle mass and $i = 1, 2$ correspond to the two cases T_1 and T_2.

Answer to question (i)

The masses contained in the generic radius r are in the two cases

$$M_i(r) = \frac{2kT_i}{G\langle m\rangle} r, \tag{9}$$

so that the ratio r_1/r_2 between the radii of the spheres containing the same amount of mass M is given by

$$\frac{r_1}{r_2} = \frac{T_2}{T_1}. \tag{10}$$

Answer to question (ii)

The period, P_i, of circular orbits of given radius R is given by $P_i = 2\pi/\omega_i$ where ω_i is the constant angular velocity, given by the centrifugal equilibrium as

$$\omega_i^2 R = G\frac{M_i(R)}{R^2} = \frac{2kT_i}{\langle m\rangle}\frac{1}{R}, \tag{11}$$

so to have

$$P_i = \frac{2\pi}{\sqrt{\frac{2kT_i}{\langle m\rangle}}} R, \tag{12}$$

leading to the searched ratio $P_1/P_2 = \sqrt{T_2/T_1}$.

SAPIENZA, UNIVERSITÁ DI ROMA
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Exercise set #4

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *April xx 2020*

Exercise 1

Determine the value of the real parameter m such that the vector $\mathbf{v} = m\,\mathbf{i} + (2-m)\,\mathbf{j} + (m-1)\mathbf{k}$ is coplanar with the vectors $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Exercise 2

Given the vector field $\mathbf{A} = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$, calculate its surface integral over the surface of the cylinder $x^2 + y^2 \leq 9$, $0 \leq z \leq 2$.

SAPIENZA, UNIVERSITÁ DI ROMA
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Exercise set #2

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *April 8 2021*

Exercise 1

Determine the expression of the mass of a semicircular wire whose density varies as the distance from the diameter joining the ends (see Fig. 1).

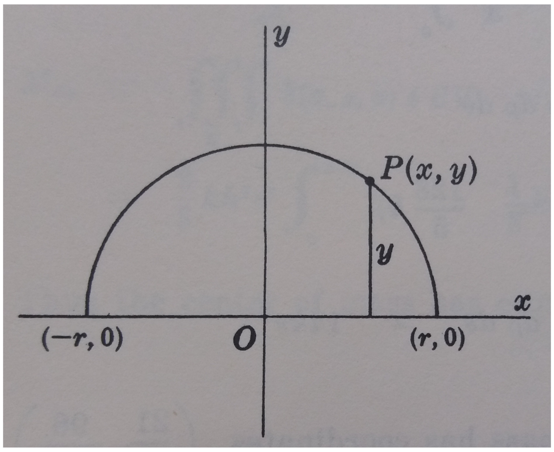

Figure 1: $P(x, y)$ is the generic point on the semicircular wire of radius r.

Exercise 2

Given a regular vector field $\mathbf{A}$ satisfying the hypotheses of the divergence theorem over normal domains containing the point $P = (x, y, z)$ in the 3D space, show that

$$\nabla \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{\int\int_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, d\sigma}{\Delta V}, \tag{1}$$

where ΔV is the volume of the normal domain enclosed by the surface ΔS, $\mathbf{n}$ is the normal unit vector outward the surface $\boldsymbol{\Delta} S$ and the limit is obtained by shrinking ΔV to the point $P = (x, y, z)$.

SAPIENZA, UNIVERSITÀ DI ROMA
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Exercise set 4

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *End of April 2022*

Exercise

Show that if $\frac{\partial \rho}{\partial \vartheta} = 0$ then $\frac{\partial U}{\partial \vartheta} = 0$, where $\rho(r, \vartheta, \varphi)$ is the mass density generating the gravitational potential $U(r, \vartheta, \varphi)$, density which is vanishing out of a generic, bound, 3D domain C where it is regular.

(Additional question left to the student: what about the case of independence of ρ on φ?)

SAPIENZA, UNIVERSITÁ DI ROMA
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Exercises

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date:

Esercizio 1

Una particella si muove in un campo in simmetria sferica e le sue coordinate polari $r \geq 0$ e $\theta \geq 0$ seguono le leggi temporali

$$\begin{cases} \dot{\theta} = \frac{2\pi}{3},\ \theta(0) = 0, \\ r(t) = R\left(1 + \left|\sin\frac{2\pi}{5}\,t\right|\right), \end{cases} \tag{1}$$

dove il punto rappresenta la derivata rispetto al tempo t e $R > 0$ è una costante.

Quesiti

1. si chiede di discutere le caratteristiche di periodicità multipla e semplice del moto;
2. le distanze di pericentro e apocentro del moto e le relative velocità.

Esercizio 2

Sia dato il potenziale

$$U(r) = \frac{U_0}{\left(\frac{r}{R} - 1\right)^2}, \tag{2}$$

per $r \geq 0$, dove $R > 0$ e $U_0 \neq 0$ sono due costanti.

Quesiti

1. discutere qualitativamente le caratteristiche del moto di una particella di massa unitaria in tale potenziale, incluse le regioni di attrattività e repulsività e l'esistenza di traiettorie circolari.

SAPIENZA, UNIVERSITÁ DI ROMA
DIP. DI FISICA

Written Exam #5 2020/2021

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *22 February 2022*

Exercise 1

Given the one-parameter family of positive spherically symmetric potentials $U(r;c)$ such to satisfy the differential equation

$$U' + \left(\frac{2}{r} - 1\right) U = 0, \tag{1}$$

where the apex stands for derivative respect to r, it is asked

Questions

(i) to determine the regions of attractivity and repulsivity of the potential;

and

(ii) to evaluate the orbital energy of the particle of unitary mass and unitary angular momentum starting from $r(0) = 1/2$ with zero radial velocity, knowing that the absolute value of the circular velocity of the unitary mass particle at distance $r = 1$ from the origin is $v_c(1) = \sqrt{2e}$ (e is the Euler's number).

Exercise 2

Two gaseous, singular, isothermal spheres in gravitational equilibrium of identical chemical composition are characterized by their (constant) temperatures T_1 and T_2.

Questions

(i) Find the ratio of the radii of the two spheres that contain the same arbitrarily given quantity of mass, M,

and

(ii) determine the ratio of the period of circular orbits of given radius R for the two cases.

SAPIENZA, UNIVERSITÁ DI ROMA
DIP. DI FISICA

Written Exam #1 2021/2022

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *15 June 2022*

Exercise 1

A particle moves subjected to a central force per unit mass

$$\mathbf{F} = \begin{cases} -Ar^{\beta}\mathbf{e}_r, \text{ for } r \leq R \\ -\dfrac{B}{r^2}\mathbf{e}_r, \text{ for } r \geq R, \end{cases} \tag{1}$$

where $A > 0$, $B > 0$, $R > 0$, β is a constant either negative, positive or null and $\mathbf{e}_r$ is the radial unit vector.

Questions

1. what are the physical dimensions of A and B?
2. what is the expression of the potential generating the force in Eq. 1?
3. what are the values of β which are compatible with a gravitational origin of the force in Eq. 1?
4. assuming $\beta > 0$, determine the intervals of energy E corresponding, respectively to i) radially periodic motion, and ii) unbound motion (the constant in the energy is determined by the Keplerian matching on the surface of the sphere of radius R);
5. assuming, again, $\beta > 0$, determine the interval of E for which radial orbits ($L_0 = 0$) are contained in the material sphere, and determine also their apocenter distance $r_+(E)$.

Exercise 2

Give an estimate of the number N of equal stars of radius = 1 $R_\odot$ uniformly distributed in a spherical cluster of radius $R = 1$ pc such that the average distance from the first neighbor is small enough to physically collide.

Note: 1 pc = 1 parsec $\simeq 3.086 \times 10^{18}$ cm; $R_\odot \simeq 6.96 \times 10^{10}$ cm

Answers

Exercise 1

1. what are the physical dimensions of A and B?

   A: $[A]L^{\beta} = LT^{-2}$ so that $[A] = \mathrm{L}^{1-\beta}\mathrm{T}^{-2}$; $[B]L^{-2} = LT^{-2}$. so that $[B]$=$\mathrm{L}^3\mathrm{T}^{-2}$;

2. what is the expression of the potential generating the force in Eq. 1?

   A: the outer ($r \geq R$) potential is obviously Keplerian,

$$U_e(r) = U_K(r) = \frac{B}{r}, \tag{2}$$

   while the inner ($r \leq R$) potential comes from integration of

$$\frac{dU_i}{dr} = -Ar^{\beta},$$

   which gives (for $\beta \neq -1$)

$$U_i(r) = -\frac{A}{\beta+1}r^{\beta+1} + c, \tag{3}$$

   and (for $\beta = -1$)

$$U_i(r) = -A\ln r + c, \tag{4}$$

   where c is a constant to be determined;

3. what are the values of β which are compatible with a gravitational origin of the force in Eq. 1?

   A: Poisson's equation leads straightforwardly to

$$\rho_{\beta}(r) = \frac{1}{4\pi G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 A r^{\beta}) = \frac{A}{4\pi G}(\beta+2)r^{\beta-1}, \tag{5}$$

   which is > 0 whenever $\beta + 2 > 0$, i.e. $\beta > -2$;

4. assuming $\beta > 0$, determine the intervals of energy E corresponding, respectively to i) radially periodic motion, and ii) unbound motion (the constant in the energy is determined by the Keplerian matching on the surface of the sphere of radius R);

   A: the constant c in the $U_i(r)$ expression of Eq. 4 is determined by letting $U_i(R) = U_K(R)$, where $U_K(r) = GM(R)/r$ is the (external) Kepler potential, where M is the mass of the sphere of radius R:

$$M \equiv M(R) = \int_0^R \rho_{\beta}(r)4\pi r^2 dr = \frac{A(\beta+2)}{G}\int_0^R r^{\beta+1}dr = \frac{A}{G}\frac{\beta+2}{\beta+2}R^{\beta+2}, \tag{6}$$

(note that the condition for the integral to converge in $r = 0$ is $\beta + 1 > -1$ which is guaranteed by the condition for ρ_β to be positive, so when $\beta > 0$ it is *a fortiori* $\beta + 1 > -1$), that leads to

$$U_K(R) = AR^{\beta+1}. \tag{7}$$

so that the condition $U_i(R) = U_K(R)$ leads to

$$-\frac{A}{\beta+1}R^{\beta+1} + c = AR^{\beta+1}, \tag{8}$$

and finally to

$$c = A\frac{\beta+2}{\beta+1}R^{\beta+1}. \tag{9}$$

Once c is known, the full expression of $V_{eff}(r;L)$ is

$$V_{eff}(r;L) = \begin{cases} \frac{1}{2}\frac{L^2}{r^2} + \frac{A}{\beta+1}R^{\beta+1}\left[\left(\frac{r}{R}\right)^{\beta+1} - (\beta+2)\right], & \text{for } r \le R, \\ \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r}, & \text{for } r \ge R. \end{cases} \tag{10}$$

There are 2 cases: $L^2 > 0$ and $L^2 = 0$ (non-radial and radial orbits). In both cases $\lim_{r\to\infty} V_{eff}(r;L) = 0^-$, while:
– if $L^2 > 0$ the above expression leads to $\lim_{r\to 0} V_{eff}(r;L) = +\infty$, because $\beta+1 > -1$ in the square parenthesis;
– if $L^2 = 0$:

$$V_{eff}(r;0) = \frac{A}{\beta+1}R^{\beta+1}\left[\left(\frac{r}{R}\right)^{\beta+1} - (\beta+2)\right], \tag{11}$$

so that

$$V_{eff}(0;0) = -\frac{A}{\beta+1}(\beta+2)R^{\beta+1}, \tag{12}$$

which – being $\beta + 2 > 0$ – is negative when $\beta + 1 > 0$ or positive when $\beta + 1 < 0$. In the present case $\beta > 0$ and so *a fortiori* it is $\beta + 1 > 0$.

On the sphere's boundary, independently of β, it is $V_{eff}(R;0) = -AR^{\beta+1} < 0$ (see plot).

If $L^2 > 0$ it is

$$V'_{eff}(r;L) = -\frac{L^2}{r^3} + Ar^\beta, \tag{13}$$

and so:

- $V'_{eff}(r;L) \le 0$. for $r^{3+\beta} \le L^2/A$, and
- $V'_{eff}(r;L) \ge 0$. for $r^{3+\beta} \ge L^2/A$,

and $r = (L^2/A)^{1/(3+\beta)}$ is the minimum point, if $(L^2/A)^{1/(3+\beta)} \leq R$ (otherwise the minimum is external to the material sphere). As a consequence:

i) radially periodic motion requires $E < 0$;
ii) unbound motion occurs for $E \geq 0$.

5. assuming, again, $\beta > 0$, determine the interval of E for which particles move on radial orbits ($L_0 = 0$) within the material sphere, and determine also their apocenter distance $r_+(E)$.

   A: The condition for radial orbits to be limited within the sphere of radius R is that. $V(0;0) \leq E \leq V(R;0)$, which reflects into

$$-A\frac{\beta+2}{\beta+1}R^{\beta+1} \leq E \leq -AR^{\beta+1}. \tag{14}$$

   The apocenter distance is given by solution of. $E - V(r;0) = 0$ that is

$$r_+(E) = R\left[\frac{(\beta+1)E}{AR^{\beta+1}} + \beta + 2\right]^{\frac{1}{\beta+1}} \tag{15}$$

Exercise 2

A: The average distance to the nearest neighbour is

$$\langle d_{nn}\rangle = \frac{1}{\sqrt[3]{n}} = \sqrt[3]{\frac{4\pi}{3}}\frac{R}{\sqrt[3]{N}}, \tag{16}$$

where N is the number of stars and n the number density, which for a uniform distribution over $r \leq R$ is

$$n = \frac{N}{\frac{4\pi}{3}R^3}. \tag{17}$$

The required condition on N to give physical collisions leads to the constraint

$$\langle d_{nn}\rangle \leq 2R_\odot, \tag{18}$$

that is

$$N \geq \frac{4\pi}{3}\left(\frac{R}{2R_\odot}\right)^3. \tag{19}$$

The exercise data give

$$N \geq \frac{4\pi}{3}\left(\frac{3.086\times10^{18}}{1.392\times10^{11}}\right)^3 \simeq 45.64\times10^{21}, \tag{20}$$

which (as expected) is an enormous number.

SAPIENZA, UNIVERSITÁ DI ROMA
DIP. DI FISICA

Written Exam #2 2021/2022

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *5 July 2022*

Exercise 1

A stellar system is characterized by a spherical density distribution

$$\rho(r) = \begin{cases} \dfrac{\rho_0}{1+\left(\frac{r}{r_c}\right)^2}, & \mathrm{r} \leq \mathrm{R}, \\ 0, & \mathrm{r} > \mathrm{R} \end{cases} \tag{1}$$

where $\rho_0 > 0$, $r_c > 0$ and $R > 0$ are three constants.

Questions

1. given that the mass contained in the sphere of radius r_c is $M(r_c) = 10^{10}\ \mathrm{M}_\odot$, and that the circular velocity at distance r_c from the center is $v_c(r_c) = 200\ \mathrm{km\ s}^{-1}$, it is asked to compute ρ_0 and r_c in $\mathrm{M}_\odot\ \mathrm{pc}^{-3}$ and in pc (parsec), respectively;
2. assuming $R = r_c$, calculate the escape velocity from the border of the sphere (adopt the proper match of inner and outer potential).

Note: 1 pc = 1 parsec $\simeq 3.086 \times 10^{18}$ cm; $\mathrm{M}_\odot \simeq \times 1.989 \times 10^{33}$ g; grav. constant $G \simeq 6.67 \times 10^{-8}\ \mathrm{cm}^3\ \mathrm{g}^{-1}\ \mathrm{s}^{-2}$.

Exercise 2

Two stellar systems (indicated by index 1 and 2) are composed by N_1 and $N_2 = 10N_1$ stars which have the same average mass, $\langle m_1 \rangle = \langle m_2 \rangle$. The stars are uniformly distributed in two spheres of same radius, $R_1 = R_2$.

Question

evaluate (formally and numerically) the ratio of the two-body relaxation times of the two systems, $t_{r,2}/t_{r,1}$.

Answers

Exercise 1

1. given that the mass contained in the sphere of radius r_c is $M(r_c) = 10^{10}\ \mathrm{M}_\odot$, and that the circular velocity at distance r_c from the center is $v_c(r_c) = 200$ km s^{-1}, it is asked to compute ρ_0 and r_c in $\mathrm{M}_\odot$ pc^{-3}, respectively;

   A: The mass contained in the sphere of radius $r \le R$ is

$$M(r) = 4\pi\rho_0 \int_0^r \frac{r^2}{1+\left(\frac{r}{r_c}\right)^2} dr = 4\pi\rho_0 r_c^3 \left(\frac{r}{r_c} - \arctan\frac{r}{r_c}\right), \tag{2}$$

   that means $M(r_c) = 10^{10}\ \mathrm{M}_\odot = 4\pi\rho_0 r_c^3(1-\pi/4)$.
   To determine ρ_0 and r_c another condition is needed, which is provided by the knowledge of the circular speed at r_c

$$v_c(r_c) = 200 \text{ km s}^{-1} = \sqrt{G\frac{M(r_c)}{r_c}} = \sqrt{4\pi G\rho_0 r_c^2\left(1-\frac{\pi}{4}\right)}. \tag{3}$$

   Combining Eqs. 2 and 3 it is easily obtained

$$r_c = G\frac{M(r_c)}{v_c^2(r_c)} \quad \text{and} \quad \rho_0 = \frac{M(r_c)}{4\pi r_c^3(1-\pi/4)}. \tag{4}$$

   Adopting the numerical values $M(r_c) = 10^{10}\mathrm{M}_\odot$ and $v(r_c) =$, it results

$$r_c \simeq 3.317\times10^{21}\text{cm} \simeq 1.075\times10^3\ \text{pc} \quad \text{and} \quad \rho_0 \simeq 2.98\,\mathrm{M}_\odot\text{pc}^{-3} \tag{5}$$

2. assuming $R = r_c$, calculate the escape velocity from the border of the sphere.

   A: The escape velocity at the border $R = r_c$ is given by $v_e(r_c) = \sqrt{2U(r_c)}$ (where $U(r)$ is the potential) which, by the Keplerian match on the border, is simply $v_e(r_c) = \sqrt{2\frac{GM(r_c)}{r_c}}$, which is equal to $\sqrt{2}v_c(r_c)$ and so (given that $v_c(r_c) = 200$ km s^{-1}) it is

$$v_e(r_c) \simeq 282,84 \text{ km s}^{-1}. \tag{6}$$

Exercise 2

Q: evaluate (formally and numerically) the ratio of the two-body relaxation times of the two systems, $t_{r,2}/t_{r,1}$.

A: The ratio of the 2 relaxation times can be expressed as

$$\frac{t_{r,2}}{t_{r,1}} = \frac{N_1 \ln N_2}{N_2 \ln N_1}\frac{t_{cr,1}}{t_{cr,2}}, \tag{7}$$

that, given that $R_1 = R_2$ and $\langle m_1\rangle = \langle m_2\rangle$ leads to

$$\frac{t_{r,2}}{t_{r,1}} = \frac{N_2 \ln N_1}{N_1 \ln N_2}\frac{\sqrt{N_1}}{\sqrt{N_2}} = \frac{\sqrt{N_2}}{\sqrt{N_1}}\frac{\ln N_1}{\ln N_2} = \sqrt{10}\frac{\ln N_1}{\ln 10 + \ln N_1}. \tag{8}$$

Given that $\ln x = \ln 10 \log_{10} x$, with a choice (for instance) of $N_1 = 1000$ (a typical open cluster) it results

$$\frac{t_{r,2}}{t_{r,1}} = \sqrt{10}\frac{\log_{10} 1000}{\log_{10} 10000} = \frac{3}{4}\sqrt{10} \simeq 2.372. \tag{9}$$

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Written Exam #5 2021/2022

Student name:

Course: *Theoretical Astrophysics* – Professor: *R. Capuzzo Dolcetta*
Due date: *13 January 2023*

Exercise 1

The circular velocity curve of a spherically symmetric galaxy is given by

$$v_c(r) = are^{-\sqrt{br}}, \tag{1}$$

where $r \geq 0$ is the distance to the galactic centre and a and b two positive constants.

Questions

1. determine the expression for the mass density of the galaxy, $\rho(r)$;
2. determine the region of positivity of $\rho(r)$;
3. verify whether the maximum of $v_c(r)$ falls in the region of positivity of $\rho(r)$;
4. assuming $b = 1 \text{ kpc}^{-1}$, determine the value of a such that the maximum value of $v_c(r)$ is equal to 320 km/s.

Note Adopt: 1 pc $=3 \times 10^{13}$ km; e = Euler's number = 2.72.

Exercise 2

Show that

$$\int_S \mathbf{n} \mathrm{d}\sigma = 0, \tag{2}$$

where S is the closed surface of a normal domain N and $\mathbf{n}$ its unit outward normal vector.

Answers

Exercise 1

1. Poisson's equation in spherical symmetry gives

$$\rho(r) = -\frac{1}{4\pi G r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right), \quad (3)$$

where $U(r)$ is the potential. The centrifugal equilibrium condition gives dU/dr as

$$\frac{dU}{dr} = -\frac{v_c^2}{r},$$

that in this case gives

$$\frac{dU}{dr} = -a^2 r e^{-2\sqrt{br}}, \quad (4)$$

and so (inserting in Eq. 3

$$\rho(r) = \frac{a^2}{4\pi G}e^{-2\sqrt{br}}\left(3 - \frac{br}{\sqrt{br}}\right). \quad (5)$$

2. Calling $\bar{r}$ the solution of $\rho(r) = 0$, it results $\rho(r) > 0$ for $r < \bar{r}$ and $\rho(r) < 0$ for $r > \bar{r}$. From Eq. 5 it results $\bar{r} = 9/b$.

3. The function $v_c(r)$ is always positive and admits a maximum in $\tilde{r}$ where $v_c'(\tilde{r}) = 0$ ($'$ denotes radial derivative). It results

$$v_c'(r) = a e^{-\sqrt{br}}\left(1 - \frac{1}{2}\sqrt{br}\right) = 0$$

for $r = \tilde{r} = 4/b < \bar{r}$.

4. The maximum circular velocity is $v_{cM} = v_c(\tilde{r}) = (4a/b)e^{-2}$. Consequently, the condition $v_{cM} = 320$ km s^{-1} leads to $a = 80be^2/(3 \times 10^{16}) \simeq 1.97 \times 10^{-14}$ s^{-1}, given the transformation from kpc to km.

Exercise 2

Making a scalar multiplication of both sides of Eq. 2 with an arbitrary constant vector $\mathbf{F}$ and using the divergence theorem it is

$$\mathbf{F} \cdot \int_S \mathbf{n} d\sigma = \int_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_N \nabla \cdot \mathbf{F} dV = 0, \quad (6)$$

as we wanted to show, given the arbitrariness of $\mathbf{F}$.